Realizations of observables in Hamiltonian systems with first class constraints.

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Abstract

In a Hamiltonian system with first class constraints observables can be defined as elements of a quotient Poisson bracket algebra. In the gauge fixing method observables form a quotient Dirac bracket algebra. We show that these two algebras are isomorphic. A new realization of the observable algebras through the original Poisson bracket is found.Generators, brackets and pointwise products of the algebras under consideration are calculated.

1. In a Hamiltonian system with the first class constraints $\varphi_j(p,q), j = 1\dots J$,

$$[\varphi_i,\varphi_j]|_{\varphi=0} = 0, \tag{1}$$

physical functions are elements of a Poisson bracket algebra of the first class functions

$$P = \{f(p,q) \mid \ [f,\varphi_j]|_{\varphi=0} = 0\}. \tag{2}$$

Here $p = (p_1,\dots,p_n), q = (q_1,\dots,q_n)$ are the canonical coordinates. Observables are elements of the algebra P/I, where

$$I = \{u(p,q) \mid \ u|_{\varphi=0} = 0\}$$

(see e.g. [1] and references therein). These definitions correspond to the Dirac quantization [2] without gauge fixing.

In the gauge fixing method [3] the gauge functions $\chi_i(p,q), i = 1\dots J$, are introduced which serve as auxiliary constraints. χ_i are supposed to satisfy the conditions

$$\det([\chi_i,\varphi_j])|_{\varphi=\chi=0} \neq 0$$

and the constraints $(\pi_\alpha) = (\varphi_1,\dots,\varphi_J,\chi_1,\dots\chi_J)$ are second class. Then the original Poisson bracket is replaced by the Dirac one

$$[g,h]_D = [g,h] - [g,\pi_\alpha]c_{\alpha\beta}[\pi_\beta,h], \qquad c_{\alpha\beta}[\pi_\beta,\pi_\gamma] = \delta_{\alpha\gamma}.$$

The constraints (π_α) are first class with respect to the Dirac bracket and physical functions are defined by the equations

$$[f,\pi_\alpha]_D|_{\pi=0} = 0$$

which are satisfied identically. Let A be the space of all the functions on phase space and

$$\Phi = \{v(p,q) \mid \ v|_{\pi=0} = 0\}.$$

The algebra of observables in the gauge fixing method is the Dirac bracket algebra A/Φ. In fact,A/Φ is a Poisson algebra, as well as P/I. A connection between the classical Hamilton equations which correspond to these two methods is described in [1].

In the present paper we show that P/I and A/Φ are isomorphic as Poisson algebras.

In a recent article [4] a family of the new algebras with respect to the original Poisson bracket was constructed which are isomorphic to a Dirac bracket algebra. The algebras which are isomorphic to A/Φ give us realizations of P/I.

Using gauge fixing functions we find a new realization of P/I. It looks like Q/K, where Q and K are Poisson subalgebras of P and I respectively.

To describe elements of P/I and prove the isomorphisms we find a local solution to equations (2). Similar equations determine elements of Q. Explicite expressions for generators enable us to calculate brackets and pointwise products for the observable algebras P/I and Q/K.

We shall assume that all the quantities vanishing on a constraint surface are linear functions of the constraints.

2. To find elements of P explicitly let us consider the equations

$$[f, \varphi_j] \in I. \tag{3}$$

with the initial condition

$$f(p, q) \in \{f_0(p, q)\}. \tag{4}$$

Here $\{f_0\} \in A/\Phi$ is the coset represented by $f_0 \in A$.

Due to (4)

$$f = f_0 + r_i \varphi_i + s_j \chi_j \tag{5}$$

for some $r_i = r_i(p, q)$, $s_j = s_j(p, q)$. Substituting this into equation (3), we get

$$[f_0, \varphi_i] + \chi_j [s_j, \varphi_i] + s_j [\chi_j, \varphi_i] \in I$$

or

$$\psi_k + (Bs)_k \in I.$$

Here

$$\psi_k = [f_0, \varphi_i] b_{ik}, \quad (Bs)_k = s_k + \chi_j [s_j, \varphi_i] b_{ik}, \quad [\chi_i, \varphi_j] b_{jk} = \delta_{ik}.$$

We assume that the operator B is locally invertible. For $u_i \in I$ we have $(B^{-1}u)_i \in I$. Hence

$$s_j = -(B^{-1}\psi)_j + s_{jk}\varphi_k \tag{6}$$

for some functions $s_{jk}(p,q)$. Expressions (5, 6) give us a solution to equations (3) with initial condition (4).

We have shown that for any $f_0 \in A$ the set $\{f_0\} \cap P$, consists of all the expressions

$$f = L(f_0) + r_i\varphi_i.$$

Here $r_i(p,q)$ are arbitrary functions and

$$L(f_0) = f_0 - \chi_j(B^{-1}\psi)_j.$$

From this it follows

$$\{f_0\} \cap P = \{L(f_0)\}_{P/I}. \tag{7}$$

Here $\{L(f_0)\}_{P/I} \in P/I$ denotes the coset represented by $L(f_0) \in P$.

To show that P/I is isomorphic to A/Φ let us define the linear function $T : P/I \to A/\Phi$

$$T(\{f\}_{P/I}) = \{f\}. \tag{8}$$

Due to (7) the inverse function $T^{-1} : A/\Phi \to P/I$ is given by

$$T^{-1}(\{f_0\}) = \{L(f_0)\}_{P/I}.$$

To show that T is a homomorphism let us compute

$$T([\{f\}_{P/I}, \{g\}_{P/I}]) = T(\{[f,g]\}_{P/I}).$$

Due to (1) for $f, g \in P$

$$[f,g] - [f,g]_D \in I.$$

From this and definition (8) it follows

$$T(\{[f,g]\}_{P/I}) = T(\{[f,g]_D\}_{P/I}) = \{[f,g]_D\} = [\{f\},\{g\}]_D = [T(\{f\}_{P/I}), T(\{g\}_{P/I})]_D.$$

Hence the Dirac bracket algebra A/Φ is isomorphic to the Poisson bracket algebra P/I.

It is easy to check that T is also an isomorphism with respect to the pointwise multiplication.We get

$$T(\{f\}_{P/I}\{g\}_{P/I}) = T(\{fg\}_{P/I}) = \{fg\} = \{f\}\{g\} = T(\{f\}_{P/I})T(\{g\}_{P/I}).$$

Thus we have shown that A/Φ and P/I are isomorphic as Poisson algebras.

3. Let us define the space

$$Q = \{F \in P \,|\, [\chi_j, F] \in I\}. \tag{9}$$

One can check that Q is a Poisson algebra and $K = Q \cap I$ is an ideal of Q.

Let F be a solution to equations (9) with the initial condition

$$F(p,q) \in \{F_0(p,q)\}_{P/I}.$$

The function F can be represented in the form

$$F = F_0 + \nu_i \varphi_i, \tag{10}$$

for some $\nu_i = \nu_i(p,q)$. Substituting (10) into equations (9) we get

$$[\chi_j, F_0] + \nu_i[\chi_j, \varphi_i] \in I.$$

A solution to these equations is

$$\nu_i = -b_{ij}[\chi_j, F_0] + \nu_{ij}\varphi_j.$$

Here $\nu_{ij} = \nu_{ij}(p,q)$ are arbitrary functions.

We have proven that for any $F_0 \in P$ the set $\{F_0\}_{P/I} \cap Q$ consists of all the expressions

$$F = R(F_0) + \nu_{ij}\varphi_i\varphi_j,$$

where

$$R(F_0) = F_0 - b_{ij}[\chi_j, F_0]\varphi_i$$

From this it follows

$$\{F_0\}_{P/I} \cap Q = \{R(F_0)\}_{Q/K}. \tag{11}$$

Here $\{R(F_0)\}_{Q/K} \in Q/K$ is the coset represented by $R(F_0) \in Q$.

Our aim is to show that Q/K is isomorphic to P/I. Let us define the linear function $S : Q/K \to P/I$

$$S(\{F\}_{Q/K}) = \{F\}_{P/I}.$$

Due to equation (11) the inverse function $S^{-1} : P/I \to Q/K$ is given by

$$S^{-1}(\{F_0\}_{P/I}) = \{R(F_0)\}_{Q/K}.$$

To show that S is a homomorphism we have the following computations

$$S(\{[F, G]\}_{Q/K}) = \{[F, G]\}_{P/I} = [\{F\}_{P/I}, \{G\}_{P/I}] = [S(\{F\}_{Q/K}), S(\{G\}_{Q/K})].$$

Hence Q/K and P/I are isomorphic as Poisson bracket algebras. It is easy to check that S is also a homomorphism with respect the pointwise multiplication

$$S(\{F\}_{Q/K}\{G\}_{Q/K}) = S(\{F\}_{Q/K})S(\{G\}_{Q/K}).$$

This tells us that Q/K and P/I are isomorphic as Poisson algebras.

Brackets and pointwise products for observables can be calculated as follows. One can check that for $f = L(f_0) \in P$ and $g = L(g_0) \in P$ the functions $[f, g]$ and fg satisfy equations (3) with the initial conditions $[f, g] \in \{[f_0, g_0]_D\}$ and $fg \in \{f_0 g_0\}$ respectively. Due to (7)

$$[f, g] = L([f_0, g_0]_D) + \tilde{u}, \quad fg = L(f_0 g_0) + \tilde{w},$$

where $\tilde{u}, \tilde{w} \in I$. From this it follows

$$[\{f\}_{P/I}, \{g\}_{P/I}] = \{L([f_0, g_0]_D)\}_{P/I},$$

$$\{f\}_{P/I}\{g\}_{P/I} = \{L(f_0 g_0)\}_{P/I}.$$

Consider the algebra Q/K. For $F = R(F_0) \in Q$ and $G = R(G_0) \in Q$ the functions $[F, G]$ and FG satisfy equations (9) with the initial conditions $[F, G] \in \{[F_0, G_0]\}_{P/I}$ and $FG \in \{F_0 G_0\}_{P/I}$ respectively. According to (11)

$$[F, G] = R([F_0, G_0]) + \tilde{U}, \quad FG = R(F_0 G_0) + \tilde{W},$$

where $\tilde{U}, \tilde{W} \in K$. Therefore, we have

$$[\{F\}_{Q/K}, \{G\}_{Q/K}] = \{R([F_0, G_0])\}_{Q/K},$$

$$\{F\}_{Q/K}\{G\}_{Q/K} = \{R(F_0 G_0)\}_{Q/K}.$$

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